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Cosmological Models

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An LRS Bianchi type II cosmological model is built with a state equation that is a function of the cosmic time t. The ratio p/μ is 1/3 when $t \to 0$ and is insignificant when $t \to \infty$. Thus, the matter content behaves like radiation for small t and like dust for large t.

1. INTRODUCTION

The discovery of the cosmic microwave radiation by Penzias and Wilson (1965), which can be interpreted as a remnant of the big-bang beginning of the universe, is a starting point for all theoretical research giving a detailed picture about how the universe evolved into its present state. One of the usual pictures is the so-called standard hot big-bang model of the universe. According to this model, our universe begins in a state of rapid expansion with an infinite density and temperature (the initial singularity). A complete thermodynamic equilibrium then holds among photons, neutrinos, electrons, hyperons, mesons, etc.

For epochs near the initial singularity we have a radiation-dominated regime where radiation pressure is 1/3 its mass-energy density,

$$P_{\text{radiation}} = \mu_{\text{radiation}}/3$$

The temperature its then redshifted by the expansion of the universe. When this falling temperature reached a few thousand degrees, the universe ceased to be radiation-dominated and became matter-dominated or nearly pressure-free (dust). In this idealized cosmology (Misner *et al.*, 1973), galaxies may be considered as particles of a gas that fills the universe. The stress-energy tensor for this fluid of galaxies is the familiar one:

$$T_{\alpha\beta} = (\mu + P) U_\alpha U_\beta + P g_{\alpha\beta}$$

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μ is the mass-energy density, P is the pressure and, U_α is the 4-vector velocity with which one must move in order to measure an isotropic intensity for the cosmic microwave radiation.

The pressure P, like the density μ, is due to both matter and radiation (Misner *et al.*, 1973):

$$P = P_{\text{matter}} + P_{\text{radiation}}, \qquad \mu = \mu_{\text{matter}} + \mu_{\text{radiation}}$$

For our universe in its present state the pressure of the matter is much less than its mass-energy density:

$$P_{\text{matter}} \ll \mu_{\text{matter}} \quad \text{(today)}$$

But the pressure of radiation is always 1/3 its mass-energy density:

$$P_{\text{radiation}} = \mu_{\text{radiation}}/3$$

Now, from current observations we have the following limits on the density of matter and radiation today (Misner *et al.*, 1973):

$$2 \times 10^{-31}\ \text{g/cm}^3 \leq \mu_{\text{matter}} \leq 10^{-28}\ \text{g/cm}^3$$

$$0.7 \times 10^{-33}\ \text{g/cm}^3 \leq \mu_{\text{radiation}} \leq 10^{-33}\ \text{g/cm}^3$$

So, pressure today is roughly 10^{-3} to 10^{-2} times its energy density μ, and with a good approximation our universe at the present epoch is dust-filled.

Considering now the analytical models that may approximately fit some of the above features characterizing our universe, it appears that most (Kramer *et al.*, 1980) of the authors working in this field choose the following procedure. They solve the Einstein field equations with a stress-energy tensor of perfect fluid type by assuming an equation of state linking the pressure P and the energy density μ, in order to build analytical models characterizing our universe near its singularity ($\mu = 3P$), or at its present epoch, where we have a matter-dominated regime.

Another alternative, which was used by Davidson (1962) and later by many others (Coley and Tupper, 1986), considers models with variable equation of state, and deals essentially with the Friedmann-Robertson-Walker metric. Here this alternative is used in the case of Bianch type II cosmologies for the first time.

The main idea used here is to leave one degree of freedom in the field equations, without assuming any equation of state and using the condition of isotropy of pressure. The result is then an analytical model with equation of state depending on the time t; once our solutions are obtained, they must satisfy the energy condition of Hawking and Ellis (1973) in order to be physically reasonable.

Once such analytical solutions are in hand, with pressure P and energy density μ linked by an equation of state such that $P \to \infty$ and $\mu \to \infty$ with $P/\mu = 1/3$ for $t \to 0$ and $P \to 0$ and $\mu \neq 0$ when $t \to \infty$, then we may confirm that they characterize the beginning of the universe or its present state.

In fact, the main characteristics of this analytical model depend on a parameter C which lies in the range $2.82842712 < C < 3$. For $C = 2.886751346$ we obtain the radiative case ($\mu = 3P$); and in this case our model describes the radiative regime (the beginning of our universe); for $8.9 \leq C^2 \leq 8.99$, the model fits the main characteristic of our universe ($P \ll \mu$); in the case $C = 3$ we approach ultimately the dust case ($P = 0$, $\mu = 0$).

In this paper, I build a homogeneous LRS Bianchi type II cosmological model (Maartens and Nel, 1978; Wainwright *et al.*, 1979; Collins, 1971, 1972; Ruban, 1978; Lorenz, 1980) satisfying all the above criteria; the distribution of matter is of perfect fluid type. This is not a restrictive case by virtue of the work of Tupper (1981, 1983*a*,*b*), which shows that the distribution of a perfect fluid may be equivalent to other complicated distributions of matter such as as viscous fluid.

2. FIELD EQUATIONS AND SOLUTION

The Einstein field equations of an LRS Bianchi type II universe filled with a perfect fluid can be expressed as follows:

$$2\frac{\dot{R}\dot{S}}{RS}+\frac{\dot{R}^2}{R^2}-\frac{S^2}{4R^4}=8\pi\mu \tag{1}$$

$$2\frac{\ddot{R}}{R}+\frac{\dot{R}^2}{R^2}-3\frac{S^2}{4R^4}=-8\pi P \tag{2}$$

$$\frac{\ddot{S}}{S}+\frac{\ddot{R}}{R}+\frac{\dot{S}\dot{R}}{SR}+\frac{S^2}{4R^4}=-8\pi P \tag{3}$$

where the dot denoted d/dt.

The elimination of the pressure P from (2) and (3) gives the condition of the isotropy of pressure,

$$\frac{\ddot{R}}{R}-\frac{\ddot{S}}{S}-\frac{\dot{S}\dot{R}}{SR}+\frac{\dot{R}^2}{R^2}-\frac{S^2}{R^4}=0 \tag{4}$$

Making the scale transformation

$$d\tau = S\,dt$$

into (4), we get

$$\frac{R''}{R}+\frac{R'^2}{R^2}-\left(\frac{S''}{S}+\frac{S'^2}{S^2}\right)-\frac{1}{R^4}=0 \tag{5}$$

where the prime denotes $d/d\tau$.

$$r = R^2, \qquad s = S^2 \tag{6}$$

we obtain

$$\frac{r''}{r} - \frac{s''}{s} - \frac{2}{r^2} = 0 \tag{7}$$

Inserting now the *ad hoc* relation

$$\frac{s''}{s} = \frac{-2}{r^2} \tag{8}$$

into (7), we obtain

$$\frac{r''}{r} = 0 \tag{9}$$

with solution

$$r(\tau) = R^2(\tau) = C\tau + C_1 \tag{10}$$

Going back now to (8), and making the scale transformation

$$t' = C\tau + C_1$$

in (8), we obtain

$$C^2 t'^2 \frac{d^2 s}{dt'^2} + 2s = 0 \tag{11}$$

The general solution of (11) is

$$s(t') = S^2(t') = A(t')^{\alpha_1} + B(t')^{\alpha_2} \tag{12}$$

where A and B are two constants of integration and α_1 and α_2 are the roots of the following equation:

$$C^2\alpha^2 - C^2\alpha + 2 = 0 \tag{13}$$

In the case $C^2 \geq 8$, α_1 and α_2 are real; note here that this solution is reduced to that of Dunn and Tupper (1980) in the case $A = 0$ or $B = 0$. (In this case the solution satisfies the equation of state $\mu = np$, where $n > 1$.)

Using the field equations (1) and (2) and the formulas (10) and (12), we obtain

$$8\pi P = \frac{A(t')^{\alpha_1 - 2}(C^2 + 3 - 2\alpha_1 C^2) + B(t')^{\alpha_2 - 2}(C^2 + 3 - 2\alpha_2 C^2)}{4} \tag{14}$$

$$8\pi\mu = \frac{A(t')^{\alpha_1 - 2}(C^2 - 1 + 2\alpha_1 C^2) + B(t')^{\alpha_2 - 2}(C^2 - 1 + 2\alpha_1 C^2)}{4} \tag{15}$$

Thus, the equation of state linking P and μ is a function of the time t'; the ratio P/μ depends on the values of C, α_1, and α_2, where

$$\tfrac{1}{2} \leq \alpha_1 = \tfrac{1}{3} + \frac{(C^2-8)^{1/2}}{2C} \leq 1 \tag{16}$$

$$0 \leq \alpha_2 = \tfrac{1}{2} - \frac{(C^2-8)^{1/2}}{2C} \leq \tfrac{1}{2} \tag{17}$$

For $C > 0$, therefore, we have

$$\frac{P}{\mu} \to \frac{C^2+3-2\alpha_2 C^2}{C^2-1+2\alpha_2 C^2} \qquad \text{for} \quad t' \to 0 \quad [(t')^{\alpha_2} \gg (t')^{\alpha_1}] \tag{18}$$

$$\frac{P}{\mu} \to \frac{C^2+3-2\alpha_1 C^2}{C^2-1+2\alpha_1 C^2} \qquad \text{for} \quad t' \to \infty \quad [(t')^{\alpha_1} \gg (t')^{\alpha_2}] \tag{19}$$

The values of P/μ are listed in the Appendix.

It is clear from the tables in the Appendix that for $C^2 = 8.33333$, this model fits the radiation-dominated regime of our universe; it behaves like a radiation-filled universe, the state equation being $\mu = 3P$. In order to describe the actual situation of our universe we must choose $8.9 < C^2 < 8.99$.

Thus, the behavior of our analytical model depends essentially on the values of C.

Finally, note that according to a theorem due to Goode and Wainwright (1986), our LRS Bianchi type II solution is of Petrov type D. Furthermore, it then appears that for suitable choices of C, our model expands without bound from an initial radiation state to reach a final dust state.

Similar results were obtained by Lemaître (1929, 1930, 1931), McIntoch (1968), May and McVittie (1970), Sistro (1971), and, more recently, Coley and Tupper (1986). Also, as far as I know, this is the first model exhibiting this type of behavior in the case of Bianchi type II cosmologies.

For A and $B > 0$ and $8.3333333 < C^2 < 9$, the strong energy conditions of Hawking and Ellis (1973),

$$-\mu \leq P_\alpha \leq \mu, \qquad \mu > 0, \qquad P > 0$$

are satisfied. Furthermore, the initial singularity is a point singularity in the sense of Thorne (1967) and MacCallum (1971).

3. CONCLUDING REMARKS

The results reached above show that using the homogeneous space-time of a Bianchi type II cosmology, one can construct analytical models within the framework of the general relativity theory which behave as radiation-filled ones for early epochs near the singularity and as dust models for the present epoch.

Table I. Values of P/μ for $t \to 0$

C^2	α_2	$C^2+3-2\alpha_2 C^2$	$C^2-1+2\alpha_2 C^2$	P/μ
8.9	0.341000319	5.830194322	13.96980568	0.417342549
8.92	0.339423685	5.86468146	13.97531854	0.419645637
8.94	0.337869333	5.898896326	13.98110367	0.421919217
8.96	0.336336583	5.932848433	13.98715157	0.424164162
8.98	0.334824784	5.966546879	13.99345312	0.42638131
8.99	0.334076556	5.983303523	13.99669648	0.427479693
9	0.333333333	6	14	0.428571428

Table II. Values of P/μ for $t \to \infty$

C^2	α_1	$C^2+3-2\alpha_1 C^2$	$C^2-1+2\alpha_1 C^2$	P/μ
8.9	0.658999681	0.169805679	19.63019432	$8.650229143 \times 10^{-3}$
8.92	0.660576315	0.135318541	19.70468146	$6.867329537 \times 10^{-3}$
8.94	0.662130667	0.101103675	19.77889633	$5.111694472 \times 10^{-3}$
8.96	0.663663417	0.087151568	19.85284843	$4.389877266 \times 10^{-3}$
8.98	0.665175216	0.033453121	19.92654688	$1.678821785 \times 10^{-3}$
8.99	0.665923444	0.016696477	19.96330352	$8.363584205 \times 10^{-4}$
9	0.66666666	0	20	0

APPENDIX

Tables I and II present the values of P/μ for various values of the model parameters.

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